

Minimal Length Effects on Entanglement Entropy of Spherically Symmetric Black Holes in Brick Wall Model

Peng Wang,^{*} Haitang Yang,[†] and Shuxuan Ying[‡]

*Center for Theoretical Physics, College of Physical Science and Technology,
Sichuan University, Chengdu, 610064, PR China*

Abstract

We compute the black hole horizon entanglement entropy for a massless scalar field in the brick wall model by incorporating the minimal length. Taking the minimal length effects on the occupation number $n(\omega, l)$ and the Hawking temperature into consideration, we obtain the leading UV divergent term and the subleading logarithmic term in the entropy. The leading divergent term scales with the horizon area. The subleading logarithmic term is the same as that in the usual brick wall model without the minimal length.

^{*}Electronic address: pengw@scu.edu.cn

[†]Electronic address: hyanga@scu.edu.cn

[‡]Electronic address: 2013222020007@stu.scu.edu.cn

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I. INTRODUCTION

Bekenstein and Hawking showed that the entropy of a black hole is proportional to the area of the horizon[1–3]. Although all the evidences suggest that the Bekenstein-Hawking entropy is truly the thermodynamic entropy, the statistical origin of the black-hole entropy has not yet been fully understood. It seems that an unavoidable candidate for the statistical origin is the entropy of the thermal atmosphere of the black hole, which can also be thought of as the entanglement entropy across the horizon[4].

To be generic, we will consider a spherically symmetric background metric of the form

$$ds^2 = f(r) dt^2 - \frac{dr^2}{f(r)} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \quad (1)$$

where $f(r)$ has a simple zero at $r = r_h$ with $f'(r_h)$ being finite and nonzero. The vanishing of $f(r)$ at point $r = r_h$ indicates the presence of an event horizon. Thus, the atmosphere entropy of the black hole with the metric (1) can be expressed in the form

$$S = \sum_{l=0}^{\infty} (2l+1) \int d\omega (\omega, l) s\left(\frac{\omega}{T}\right), \quad (2)$$

where ω is the Killing energy associated with t , l is the angular momentum, T is the Hawking temperature of the black hole, $n(\omega, l)$ is the number of one-particle states not exceeding ω with fixed value of angular momentum l , and $s\left(\frac{\omega}{T}\right)$ is the thermal entropy per mode. However, the entropy diverges when we attempt to sum eqn. (2) over all the modes. There are two kinds of divergences. The first one is due to infinite volume of the system, which has to do with the contribution from the vacuum surrounding the system at large distances and

is of little relevance here. The second one arises from the infinite volume of the deep throat region near the horizon. In order to regulate the two divergences, t' Hooft [5] proposed the brick wall model for a scalar field ϕ , where two brick wall cutoffs are introduced at some small distance r_ε from the horizon and at a large distance $L \gg r_h$,

$$\phi = 0 \quad \text{at } r = r_h + r_\varepsilon \text{ and } r = L. \quad (3)$$

The minimally coupled scalar field satisfies the Klein-Gordon equation

$$\left(\frac{\nabla^2}{\hbar^2} - m^2 \right) \phi = 0. \quad (4)$$

Since WKB approximation is reliable as long as the black hole's mass $M \gg 1$ (in Planck units), t' Hooft took the ansatz for ϕ

$$\phi = \exp \left[-\frac{i}{\hbar} \omega t + \frac{i}{\hbar} \int p_r dr \right] Y_{lm}(\theta, \phi), \quad (5)$$

and solved for p_r to the WKB leading order. Define the radial wave number $k(r, l, \omega)$ by

$$k(r, l, \omega) = |p_r|, \quad (6)$$

as long as $p_r^2 \geq 0$, and $k(r, l, \omega) = 0$ otherwise. With the two Dirichlet boundaries (3), $n(\omega, l)$ can be expressed as

$$n(\omega, l) = \frac{1}{\pi \hbar} \int_{r_h + r_\varepsilon}^L k(r, l, \omega) dr. \quad (7)$$

It would appear that the entanglement entropy is sensitive to the ultraviolet(UV) behavior of quantum fields, where quantum gravity effects become important. Even though there is still no complete and consistent quantum theory of gravity, one could still rely on effective models to study the UV behavior of the entanglement entropy. For example, this issue was studied in [6, 7] in the context of the possible modifications of the standard dispersion relation. On the other hand, various theories of quantum gravity, such as string theory, loop quantum gravity and quantum geometry, predict the existence of a minimal length[8–10]. The generalized uncertainty principle(GUP)[11] is a simple way to realize this minimal length. In [12–15], the authors considered the generalized uncertainty relation

$$\Delta x \Delta p \geq \hbar + \frac{\lambda}{\hbar} (\Delta p)^2, \quad (8)$$

which gives a minimal length, $2\sqrt{\lambda}$. As a consequence, the number of quantum states should be changed to

$$\frac{d^3x d^3p}{(2\pi\hbar)^3 (1 + \lambda p^2)^3}, \quad (9)$$

where $p^2 = p_i p^i$. Therefore, they had for massless particles

$$n(\omega, l) = \frac{1}{\pi\hbar} \int_{r_h+r_\varepsilon}^L \frac{k(r, l, \omega)}{(1 + \lambda\omega^2/f)^3} dr, \quad (10)$$

where $k(r, l, \omega)$ is given by eqn. (6). The all order generalized uncertainty relation was also considered in [16]. It is found there that the artificial cutoff parameter in the brick wall model located just outside the horizon can be avoided if GUP is considered. Alternatively, the modified Klein-Gordon equation for ϕ was considered in the framework of Horava-Lifshitz gravity and GUP in [17]. The WKB leading term of p_r for the modified Klein-Gordon equation was obtained and $n(\omega, l)$ was given by eqn. (7). The entanglement entropy was then calculated and the result is consistent with previous studies.

Furthermore, GUP should modify the Hawking temperature T in eqn. (2) as well as $n(\omega, l)$. Indeed, the GUP deformed Hamilton-Jacobi equation in curved spacetime have been introduced and the corrected Hawking temperatures have been derived in [18–22]. The GUP corrections to the Hawking temperature were found to depend not only on the black hole's mass but the mass, the energy, and the angular momentum of emitted particles as well. In [22], we derived the deformed Klein-Gordon and Dirac equation incorporating the GUP form proposed in [23, 24]. The GUP modified Hawking temperatures for scalars and fermions were then obtained and the black hole's evaporation was also discussed in [22]. In this paper, we will investigate the entropy of the thermal atmosphere of the black hole in the context of [22]. Taking GUP corrections to both $n(\omega, l)$ and Hawking temperature T into consideration, we will calculate the entanglement entropy of a massless scalar field via eqn. (2) in the brick wall model.

The organization of this paper is as follows. In section II, we review some results of [22], which are necessary for calculating the entanglement entropy. In section III, the entanglement entropy of a massless scalar field near the horizon is calculated in the brick wall model. Section IV is devoted to our discussion and conclusion. In this paper, we take Geometrized units $c = G = 1$, where the Planck constant \hbar is square of the Planck Mass m_p . For simplicity, we assume that the emitted particles are massless and neutral.

II. DEFORMED HAMILTON-JACOBI METHOD

In [22] and this paper, we consider an effective model of the GUP in one dimensional quantum mechanics given by[23, 24]

$$L_f k(p) = \tanh\left(\frac{p}{M_f}\right), \quad (11)$$

$$L_f \omega(E) = \tanh\left(\frac{E}{M_f}\right), \quad (12)$$

where the generators of the translations in space and time are the wave vector k and the frequency ω , L_f is the minimal length, and $L_f M_f = \hbar$. From eqn. (11), it is noted that although one can increase p arbitrarily, k has an upper bound which is $\frac{1}{L_f}$. The upper bound on k implies that particles could not possess arbitrarily small Compton wavelength $\lambda = 2\pi/k$ and that there exists a minimal length $\sim L_f$. The quantization in position representation $\hat{x} = x$ leads to

$$k = -i\partial_x, \quad \omega = i\partial_t. \quad (13)$$

In the $(3+1)$ dimensional flat spacetime, the relations between (p_i, E) and (k_i, ω) can simply be generalized to

$$L_f k_i(p) = \tanh\left(\frac{p_i}{M_f}\right), \quad (14)$$

$$L_f \omega(E) = \tanh\left(\frac{E}{M_f}\right), \quad (15)$$

where one has for \vec{k} in the spherical coordinates

$$\vec{k} = -i \left(\hat{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r \sin \theta} \frac{\partial}{\partial \phi} \right). \quad (16)$$

Expanding eqn. (14) and eqn. (15) for small arguments to the third order gives the energy and momentum operator in position representation

$$E = i\hbar\partial_t \left(1 - \frac{\hbar^2}{M_f^2} \partial_t^2 \right), \quad (17)$$

$$\vec{p} = \frac{\hbar}{i} \left[\hat{r} \left(\partial_r - \frac{\hbar^2 \partial_r^3}{M_f^2} \right) + \hat{\theta} \left(\frac{\partial_\theta}{r} - \frac{\hbar^2 \partial_\theta^3}{r^3 M_f^2} \right) + \hat{\phi} \left(\frac{\partial_\phi}{r \sin \theta} - \frac{\hbar^2 \partial_\phi^3}{r^3 \sin^3 \theta M_f^2} \right) \right], \quad (18)$$

where we also omit the factor $\frac{1}{3}$. Substituting the above energy and momentum operators into the energy-momentum relation, the deformed Klein-Gordon equation satisfied by the

massless scalar field is

$$E^2 \phi = p^2 \phi, \quad (19)$$

where $p^2 = \vec{p} \cdot \vec{p}$. Making the ansatz for ϕ

$$\phi = \exp\left(\frac{iI}{\hbar}\right), \quad (20)$$

and substituting it into eqn. (19), one expands eqn. (19) in powers of \hbar and then finds that the lowest order gives the deformed scalar Hamilton-Jacobi equation in the flat spacetime

$$\begin{aligned} (\partial_t I)^2 \left(1 + \frac{2(\partial_t I)^2}{M_f^2}\right) - (\partial_r I)^2 \left(1 + \frac{2(\partial_r I)^2}{M_f^2}\right) - \frac{(\partial_\theta I)^2}{r^2} \left(1 + \frac{2(\partial_\theta I)^2}{r^2 M_f^2}\right) \\ - \frac{(\partial_\phi I)^2}{r^2 \sin^2 \theta} \left(1 + \frac{2(\partial_\phi I)^2}{r^2 \sin^2 \theta M_f^2}\right) = 0, \end{aligned} \quad (21)$$

which is truncated at $\mathcal{O}\left(\frac{1}{M_f^2}\right)$. The Hamilton-Jacobi equation in the metric (1) can be obtained from that in flat spacetime by making replacements $\partial_r I \rightarrow \sqrt{f(r)} \partial_r I$ and $\partial_t I \rightarrow \frac{\partial_t I}{\sqrt{f(r)}}$. Therefore, the deformed Hamilton-Jacobi equation in flat spacetime, eqn. (21), leads to the deformed Hamilton-Jacobi equation in the metric (1), which is to $\mathcal{O}\left(\frac{1}{M_f^2}\right)$

$$\begin{aligned} \frac{(\partial_t I)^2}{f(r)} \left(1 + \frac{2(\partial_t I)^2}{f(r) M_f^2}\right) - f(r) (\partial_r I)^2 \left(1 + \frac{2f(r) (\partial_r I)^2}{M_f^2}\right) - \frac{(\partial_\theta I)^2}{r^2} \left(1 + \frac{2(\partial_\theta I)^2}{r^2 M_f^2}\right) \\ - \frac{(\partial_\phi I)^2}{r^2 \sin^2 \theta} \left(1 + \frac{2(\partial_\phi I)^2}{r^2 \sin^2 \theta M_f^2}\right) = 0. \end{aligned} \quad (22)$$

Taking into account the Killing vectors of the background spacetime, we can employ the following ansatz for the action

$$I = -\omega t + W(r, \theta) + p_\phi \phi, \quad (23)$$

where ω and p_ϕ are constants and they are the energy and the z -component of angular momentum of emitted particles, respectively. Inserting eqn. (23) into eqn. (22), we find that the deformed Hamilton-Jacobi equation becomes

$$p_r^2 \left(1 + \frac{2f(r) p_r^2}{M_f^2}\right) = \frac{1}{f^2(r)} \left[\omega^2 \left(1 + \frac{2\omega^2}{f(r) M_f^2}\right) - f(r) \frac{L^2}{r^2} \right], \quad (24)$$

where we neglect terms higher than $\mathcal{O}\left(\frac{1}{M_f^2}\right)$. In the above equation, we define $p_r = \partial_r W$, $p_\theta = \partial_\theta W$, $L^2 = p_\theta^2 + \frac{p_\phi^2}{\sin^2 \theta}$, and $L^2 = (l+1)l\hbar^2$ which is the magnitude of the angular

momentum of the particle. Solving eqn. (24) for p_r to $\mathcal{O}\left(\frac{1}{M_f^2}\right)$ gives

$$p_r^\pm = \pm \left(\frac{\Omega^{\frac{1}{2}}}{f(r)} - \frac{\Omega^{\frac{3}{2}}}{f^2(r) M_f^2} \right), \quad (25)$$

where $+/-$ represent the outgoing/ingoing solutions and we define

$$\Omega = \omega^2 \left(1 + \frac{2\omega^2}{f(r) M_f^2} \right) - f(r) \frac{(l+1) l \hbar^2}{r^2}. \quad (26)$$

Using the residue theory for semi circles, we obtain for the imaginary part of W_\pm to $\mathcal{O}\left(\frac{1}{M_f^2}\right)$

$$\text{Im } W_\pm = \pm \frac{\pi\omega}{f'(r_h)} \left[1 + \frac{2(l+1) l \hbar^2}{M_f^2 r_h^2} \right]. \quad (27)$$

Taking both the spatial contribution and the temporal contribution into account[25–29], one finds that the tunnelling rate of the particle crossing the horizon is

$$\begin{aligned} \Gamma &\propto \exp \left[-\frac{1}{\hbar} \left(\text{Im}(\omega \Delta t) + \text{Im} \oint p_r dr \right) \right] \\ &= \exp \left\{ -\frac{4\pi\omega}{f'(r_h)} \left[1 + \frac{2(l+1) l \hbar^2}{M_f^2 r_h^2} \right] \right\}. \end{aligned} \quad (28)$$

This is the expression of Boltzmann factor with an effective temperature to $\mathcal{O}\left(\frac{1}{M_f^2}\right)$

$$T = \frac{T_0}{1 + \Delta}, \quad (29)$$

where $T_0 = \frac{\hbar f'(r_h)}{4\pi}$ is the original Hawking temperature and we define

$$\Delta = \frac{(l+1) l \hbar^2}{M_f^2 r_h^2}. \quad (30)$$

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III. ENTANGLEMENT ENTROPY OF BLACK HOLE

For scalar particles emitted in a wave mode labelled by energy ω and l , we find from eqn. (28) that [30]

$$\begin{aligned} &(\text{Probability for a black hole to emit a particle in this mode}) \\ &= \exp \left(-\frac{\omega}{T} \right) \times (\text{Probability for a black hole to absorb a particle in the same mode}), \end{aligned}$$

where T is given by eqn. (29). Neglecting back-reaction, detailed balance condition requires that the ratio of the probability of having N particles in a particular mode with ω and l to the probability of having $N - 1$ particles in the same mode is $\exp\left(-\frac{\omega}{T}\right)$. One then follows the standard textbook procedure to get the average number $n_{\omega,l}$ in the mode

$$n_{\omega,l} = n\left(\frac{\omega}{T}\right), \quad (31)$$

where we define

$$n(x) = \frac{1}{\exp x - 1}. \quad (32)$$

The von Neumann entropy for the mode is

$$s_{\omega,l} = (n_{\omega,L} + 1) \ln(1 + n_{\omega,L}) - n_{\omega,L} \ln n_{\omega,L}, \quad (33)$$

where $s_{\omega,l}$ is $s\left(\frac{\omega}{T}\right)$ in eqn. (2). The $s(x)$ is given by

$$s(x) = \frac{\exp x}{\exp x - 1} \ln\left(\frac{\exp x}{\exp x - 1}\right) + \frac{\ln(\exp x - 1)}{\exp x - 1}. \quad (34)$$

Define the radial wave number $k(r, l, \omega)$ by

$$k(r, l, \omega) = |p_r^\pm|, \quad (35)$$

as long as $p_r^2 \geq 0$, and $k(r, l, \omega) = 0$ otherwise. The p_r^\pm is given by eqn. (25). Taking two Dirichlet conditions at $r = r_h + r_\varepsilon$ and $r = L$ in the brick wall model into account, we find that the number of one-particle states not exceeding ω with fixed value of angular momentum l is given by

$$n(\omega, l) = \frac{1}{\pi \hbar} \int_{r_h + r_\varepsilon}^L k(r, l, \omega) dr. \quad (36)$$

Thus, we get for the total entropy of radiation

$$\begin{aligned} S &= \sum_{\omega, l, m} s_{\omega, l} = \int (2l + 1) dl \int d\omega \frac{dn(\omega, l)}{d\omega} s_{\omega, l} \\ &= -\frac{1}{\pi \hbar^3} \int d[(l + 1) l \hbar^2] \int d\omega \frac{\partial s_{\omega, l}}{\partial \omega} \int_{r_h + r_\varepsilon}^L dr k(r, l, \omega). \end{aligned} \quad (37)$$

Defining $u = \frac{\omega}{T_0}$ and expanding $s_{\omega, l}$ to $\mathcal{O}\left(\frac{1}{M_f^2}\right)$

$$s_{\omega, l} \approx s(u) + s'(u) u \Delta, \quad (38)$$

we find that the entropy to $\mathcal{O}\left(\frac{1}{M_f^2}\right)$ is

$$S \approx -\frac{M_f^2 r_h^2}{\pi \hbar^3} \int du \int_{r_h+r_\varepsilon}^L dr \int d\Delta [s'(u) + (s'(u)u)'\Delta] \left(\frac{\Omega^{\frac{1}{2}}}{f(r)} - \frac{\Omega^{\frac{3}{2}}}{f^2(r) M_f^2} \right),$$

where we use eqn. (30) for Δ and eqn. (25) for $k(r, l, \omega)$. Performing Δ integral which runs over the region where $\Omega > 0$ gives

$$S \approx \frac{2r_h^2 T_0^3}{\pi \hbar^3} \int s(u) du \int_{r_h+r_\varepsilon}^L dr \frac{r^2}{r_h^2} \frac{u^2}{f^2(r)} \left[1 + \frac{4T_0^2 u^2}{f(r) M_f^2} - \frac{6T_0^2 u^2}{5f(r) M_f^2} \frac{r^2}{r_h^2} \right], \quad (39)$$

where the second term in the bracket comes from the GUP corrections to $n(\omega, l)$ and the third term from the GUP corrections to Hawking temperature T . Focusing on the divergent parts near horizon, we find

$$\begin{aligned} \int_{r_h+r_\varepsilon}^L dr \frac{r^2}{r_h^2} \frac{1}{f^2(r)} &\sim \frac{1}{4\kappa^2 r_\varepsilon} - \left(\frac{1}{\kappa r_h} - \eta \right) \frac{\ln \kappa r_\varepsilon}{2\kappa}, \\ \int_{r_h+r_\varepsilon}^L dr \frac{r^2}{r_h^2} \frac{1}{f^3(r)} &\sim \frac{1}{4\kappa^3 r_\varepsilon^2} + \frac{\frac{2}{\kappa r_h} - 3\eta}{8\kappa^2 r_\varepsilon} - \left(\frac{1}{\kappa^2 r_h^2} + 3(2\eta^2 - \theta) - \frac{6\eta}{\kappa r_h} \right) \frac{\ln \kappa r_\varepsilon}{8\kappa}, \\ \int_{r_h+r_\varepsilon}^L dr \frac{r^4}{r_h^4} \frac{1}{f^3(r)} &\sim \frac{1}{4\kappa^3 r_\varepsilon^2} + \frac{\frac{4}{\kappa r_h} - 3\eta}{8\kappa^2 r_\varepsilon} - \left(\frac{6}{\kappa^2 r_h^2} + 3(2\eta^2 - \theta) - \frac{12\eta}{\kappa r_h} \right) \frac{\ln \kappa r_\varepsilon}{8\kappa}, \end{aligned} \quad (40)$$

where we expand $f(r)$ and $\frac{r^2}{r_h^2}$ at $r = r_h$

$$\begin{aligned} f(r) &\sim 2\kappa(r - r_h) [1 + \eta\kappa(r - r_h) + \theta\kappa^2(r - r_h)^2], \\ \frac{r^2}{r_h^2} &\sim 1 + \frac{2\kappa(r - r_h)}{\kappa r_h} + \frac{\kappa^2(r - r_h)^2}{\kappa^2 r_h^2}. \end{aligned} \quad (41)$$

In eqn. (40), we neglect finite terms as $\kappa r_\varepsilon \rightarrow 0$ and terms involving L . Note that we define $\kappa = \frac{f'(r_h)}{2}$ which is the surface gravity for the black hole and hence $T_0 = \frac{\hbar\kappa}{2\pi}$. Thus, the divergent part of entropy near the horizon is

$$\begin{aligned} S &\sim \frac{2r_h^2 T_0^3}{\pi \hbar^3 \kappa} \int u^2 s(u) du \left\{ \frac{1}{4\kappa r_\varepsilon} - \left(\frac{1}{\kappa r_h} - \eta \right) \frac{\ln \kappa r_\varepsilon}{2} + \right. \\ &\quad \left. \frac{T_0^2 u^2}{M_f^2} \left[\frac{14}{20} \frac{1}{\kappa^2 r_\varepsilon^2} + \frac{\frac{8}{\kappa r_h} - 21\eta}{20\kappa r_\varepsilon} + \frac{2}{5} \left(\frac{28}{\kappa^2 r_h^2} + 39(2\eta^2 - \theta) - \frac{96\eta}{\kappa r_h} \right) \frac{\ln \kappa r_\varepsilon}{8} \right] \right\}. \end{aligned} \quad (42)$$

Defining the proper distance ε between the brick wall and the horizon

$$\varepsilon = \int_{r_h}^{r_h+r_\varepsilon} \frac{dr}{\sqrt{f}} \sim \frac{\sqrt{r_\varepsilon}}{\sqrt{2\kappa}} \left(2 - \frac{\eta\kappa r_\varepsilon}{3} \right),$$

one could express eqn. (42) in terms of ε

$$S \sim \frac{r_h^2}{8\pi^4} \int u^2 s(u) du \left\{ \frac{1}{\varepsilon^2} - 2\kappa^2 \left(\frac{1}{\kappa r_h} - \eta \right) \ln \kappa \varepsilon \right. \\ \left. + \frac{L_f^2 u^2 \kappa^2}{2\pi^2 \varepsilon^2} \left[\frac{14}{5\kappa^2 \varepsilon^2} + \frac{2 \left(5\eta - \frac{39}{\kappa r_h} \right)}{45} + \frac{2\kappa^2 \varepsilon^2}{5} \left(\frac{28}{\kappa^2 r_h^2} + 39(2\eta^2 - \theta) - \frac{96\eta}{\kappa r_h} \right) \frac{\ln \kappa \varepsilon}{4} \right] \right\}, \quad (43)$$

where we use $L_f = \frac{\hbar}{M_f}$. A natural choice for ε is the minimal length L_f . If we take $\varepsilon = L_f$, we have for the entanglement entropy near the horizon

$$S \sim \frac{r_h^2}{8\pi^4 L_f^2} \int_0^\infty \left(1 + \frac{14u^2}{10\pi^2} \right) u^2 s(u) du \\ - \frac{r_h \kappa}{4\pi^4} (1 - \eta \kappa r_h) \int u^2 s(u) du \ln \kappa L_f + \text{Finite terms as } \varepsilon L_f \rightarrow 0 \\ \sim \frac{17A}{1800\pi L_f^2} - \frac{r_h \kappa (1 - \eta \kappa r_h)}{45} \ln \kappa L_f + \text{Finite terms as } \varepsilon L_f \rightarrow 0, \quad (44)$$

where $A = 4\pi r_h^2$ is the horizon area.

IV. DISCUSSION AND CONCLUSION

In [5, 31], the entanglement entropy of a massless scalar field near the horizon of the metric (1) was calculated in the brick wall model without the GUP. It has been shown the entropy in this case is

$$S \sim \frac{A}{360\pi \varepsilon^2} - \frac{\kappa r_h}{45} (1 - \kappa \eta r_h) \ln \kappa \varepsilon, \quad (45)$$

where κ and η are defined in eqn. (41) and ε is the proper distance between the wall and the horizon. In our paper, we take $\varepsilon = L_f$ which is the minimal length. The numerical factor of the leading term of the entropy in eqn. (44) has been changed from $\frac{1}{360\pi}$ to $\frac{17}{1800\pi}$ due to the minimal length effects. The leading terms of eqn. (44) and eqn. (45) are both proportional to the horizon area. The area law in the brick wall model was first obtained in [5] and later was also studied in [32–35]. In [12] where the GUP form (8) was used, the leading term of the entanglement entropy for the spherical metric (1) was calculated up to $\mathcal{O}(\lambda)$. The result was

$$S \sim \frac{3}{L_f^2 \pi} \frac{A}{4}, \quad (46)$$

where $L_f = 2\sqrt{\lambda}$ is the minimal length. In [16] where the all order GUP form

$$\Delta x \Delta p \geq \frac{\hbar}{2} \exp \left[\frac{\lambda}{\hbar^2} (\Delta p)^2 \right], \quad (47)$$

was considered, the leading term of the all order entanglement entropy for the spherically symmetric metric (1) was given by

$$S \sim \frac{e^3 \zeta(3)}{4\pi L_f^2} \frac{A}{4}, \quad (48)$$

where $L_f = \sqrt{\frac{e\lambda}{2}}$ and $\zeta(3) = \sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.202$. In both cases, the area law in the brick wall model is preserved after the GUP corrections to $n(\omega, l)$ are considered. Our result shows that the area law is still valid after the GUP corrections to both $n(\omega, l)$ and the Hawking temperature T are included. If we choose the invariant cutoff $\varepsilon = \frac{m_p}{\sqrt{90\pi}}$ in t' Hooft's original calculation, then the dominant entropy term becomes the Bekenstein-Hawking entropy, $S \sim S_{BH} \equiv \frac{A}{4m_p^2}$. Similarly, if we have $L_f = \sqrt{\frac{17}{450\pi}} m_p \approx 0.1 m_p$ in our calculation, S_{BH} also appears for the leading entropy term. On the other hand, it is normally assumed that the minimal L_f is of the order of m_p . If $L_f \sim m_p$, one finds the entanglement entropy for one scalar field $S \sim 10^{-2} S_{BH}$. Since the entanglement entropy depends on the number and nature of quantum fields while S_{BH} is universal, the ‘‘species problem’’ arises when the the entanglement entropy makes important contributions to S_{BH} . Taking $L_f \sim m_p$ would provide a resolution to the species problem.

The subleading logarithmic part of the entanglement entropy is also calculated in our paper. It turns out that the subleading logarithmic part is universally given by $-\frac{\kappa r_h}{45} (1 - \kappa \eta r_h) \ln \kappa \varepsilon$ in scenarios both with and without the GUP. For the Schwarzschild metric with $f(r) = 1 - \frac{2M}{r}$, the subleading logarithmic part becomes $-\frac{\ln \kappa \varepsilon}{45}$, which was first obtained in [36]. On the other hand, we can estimate the entropy of the black hole using eqn. (29). In fact, the angular momentum of the particle $L \sim pr_h \sim \omega r_h$ near the horizon of the the black hole. Thus, one can rewrite T

$$T \sim \frac{T_0}{1 + \frac{2\omega^2}{M_f^2}}, \quad (49)$$

where $T_0 = \frac{\hbar}{8\pi M}$ for the Schwarzschild black hole. As reported in [37], the authors obtained the relation $\omega \gtrsim \frac{\hbar}{\delta x}$ between the energy of a particle and its position uncertainty in the framework of GUP. Near the horizon of the the Schwarzschild black hole, the position uncertainty of a particle will be of the order of the Schwarzschild radius of the black hole [38] $\delta x \sim r_h$. Thus, one finds for T

$$T \sim \frac{T_0}{1 + \frac{m_p^4}{2M^2 M_f^2}}, \quad (50)$$

where we use $\hbar = m_p^2$. Using the first law of the black hole thermodynamics, we find the corrected black hole entropy is

$$S = \int \frac{dM}{T} \sim \frac{A}{4m_p^2} + \frac{4\pi L_f^2}{m_p^2} \ln \left(\frac{A}{16\pi} \right), \quad (51)$$

where $A = 4\pi r_h^2 = 16\pi M^2$ is the area of the horizon. The logarithmic term in eqn. (51) is the well known correction from quantum gravity to the classical Bekenstein-Hawking entropy, which have appeared in different studies of GUP modified thermodynamics of black holes[39–41]. Comparing eqn. (44) with eqn. (51), we note that there are two discrepancies in the subleading logarithmic term, one of which is the sign and the other dependence on the minimal length L_f . These discrepancies would imply that the entanglement entropy could not solely account for the entropy of the black hole.

It has been shown in [12–16] that the artificial cutoff parameter in the brick wall model located just outside the horizon can be avoided if GUP is considered. However, this is not the case in our paper. Actually, if one attempts to let $r_\epsilon = 0$ in eqn. (42), the entropy diverges and a brick wall is still needed. How can we reconcile the contradiction? One might note that we calculate the entanglement entropy to $\mathcal{O}\left(\frac{1}{M_f^2}\right)$. For the typical energy $\omega \sim T_0 = \frac{\hbar\kappa}{2\pi}$, one finds that the $\mathcal{O}\left(\frac{1}{M_f^2}\right)$ GUP corrections to the entropy $\sim \frac{\omega^2}{f(r)M_f^2} \sim \frac{\hbar^2\kappa^2}{4\pi^2 f(r)M_f^2}$. At the wall at $r_\epsilon = r_h + 2\kappa L_f^2$, we have $\frac{\hbar^2\kappa^2}{f(r_\epsilon)M_f^2} \sim \frac{\hbar^2\kappa^2}{16\pi^2\kappa^2 L_f^2 M_f^2} \sim \frac{1}{16\pi^2}$. Thus, our perturbative method is valid outside the wall at $r_\epsilon = r_h + 2\kappa L_f^2$. However, the perturbation would break down deep within the wall and one needs to consider higher order contributions. The divergence of the entropy eqn. (42) as $r_\epsilon \rightarrow 0$ is more like due to the breaking down of our perturbative method. In [12–15], it is crucial for $\lambda\omega^2/f$ to be in the denominator of the integrand in eqn. (10) to get rid of the wall. If one replaces $(1 + \lambda\omega^2/f)^{-3}$ with $1 - 3\lambda\omega^2/f$ in eqn. (10), the integral diverges as $r \rightarrow r_h$. The divergence arises simply because the Taylor expansion of $(1 + \lambda\omega^2/f)^{-3}$ breaks down when $\lambda\omega^2/f > 1$.

In this paper, we calculate the entanglement entropy of a massless scalar field near the horizon of a 4D spherically symmetric black hole in the brick wall model incorporating the minimal length effects. We show that the leading term of the entropy is proportional to the horizon area. If the minimal length $L_f \sim m_p$, the entanglement entropy makes a small contribution to the Bekenstein-Hawking entropy, which might resolve the “species problem”. The subleading logarithmic term is also calculated. The result is the same as the one in the usual brick wall model without the minimal length and independent of the minimal length

L_f .

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